



TITLE:

# ON ABSOLUTE NORMS ON $C^2$ : the Geometric Aspect (Nonlinear Analysis and Convex Analysis)

AUTHOR(S):

Kato, Mikio; Saito, Kichi-Suke; Takahashi, Yasuji

---

CITATION:

Kato, Mikio ...[et al]. ON ABSOLUTE NORMS ON  $C^2$ : the Geometric Aspect (Nonlinear Analysis and Convex Analysis). 数理解析研究所講義録 2000, 1136: 133-138

ISSUE DATE:

2000-04

URL:

<http://hdl.handle.net/2433/63775>

RIGHT:

# ON ABSOLUTE NORMS ON $\mathbb{C}^2$

## —the Geometric Aspect

Mikio Kato (加藤幹雄)\*, Kichi-Suke Saito (斎藤吉助)<sup>†</sup>  
and Yasuji Takahashi (高橋泰嗣)<sup>†</sup>

This is an announcement of some recent results of the authors [10, 11]. Each absolute normalized norm on  $\mathbb{C}^2$  corresponds to a continuous convex function on  $[0, 1]$  satisfying certain conditions (cf. [2]). By virtue of this correspondence we can obtain many concrete norms of non  $\ell_p$ -type on  $\mathbb{C}^2$ . In this note we discuss some geometrical properties of absolute norms by means of the corresponding convex functions. Also we consider the direct sums of Banach spaces with norms associated with these functions.

### 1. Absolute norms on $\mathbb{C}^2$

A norm  $\|\cdot\|$  on  $\mathbb{C}^2$  is called *absolute* if

$$(1) \quad \|(z, w)\| = \|(|z|, |w|)\| \quad \forall z, w \in \mathbb{C},$$

and is called *normalized* if  $\|(1, 0)\| = \|(0, 1)\| = 1$ . The  $\ell_p$ -norms  $\|\cdot\|_p$  are typical such examples. Let  $N_a$  denote the family of all absolute normalized norms on  $\mathbb{C}^2$ .

**Lemma A** (Bonsall-Duncan [2]). *For any  $\|\cdot\| \in N_a$*

$$(2) \quad \|\cdot\|_\infty \leq \|\cdot\| \leq \|\cdot\|_1.$$

Indeed, for any  $(z, w) \in \mathbb{C}^2$

$$\begin{aligned} \|(z, w)\|_\infty &= \max\{\|(z, 0)\|, \|(0, w)\|\} \\ &\leq \frac{1}{2} \max\{\|(z, w)\| + \|(z, -w)\|, \|(z, w)\| + \|(-z, w)\|\} \\ &= \|(z, w)\| \\ &\leq \|(z, 0)\| + \|(0, w)\| \\ &= \|(z, w)\|_1. \end{aligned}$$

**Theorem A** (Bonsall-Duncan [2]). (i) *For any norm  $\|\cdot\| \in N_a$  put*

$$(3) \quad \psi(t) = \|(1-t, t)\| \quad (0 \leq t \leq 1).$$

Then  $\psi$  is continuous and convex on  $[0, 1]$ , and

$$(4) \quad \psi(0) = \psi(1) = 1, \quad \max\{1-t, t\} \leq \psi(t) \leq 1.$$

(ii) For any  $\psi \in \Psi$  let

$$(5) \quad \|(z, w)\|_\psi = \begin{cases} (|z| + |w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & ((z, w) \neq (0, 0)), \\ 0 & ((z, w) = (0, 0)). \end{cases}$$

Then  $\|\cdot\|_\psi \in N_a$  and  $\|\cdot\|_\psi$  satisfies (3).

Put

$$(6) \quad \Psi := \{\psi : \text{continuous convex function on } [0, 1] \text{ with (4)}\}.$$

According to Theorem A,  $N_a$  and  $\Psi$  are in 1-1 correspondence under (3). For instance the  $\ell_p$ -norm  $\|\cdot\|_p$  is associated with

$$\psi_p(t) := \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty. \end{cases}$$

## 2. Von Neumann-Jordan constant

The *von Neumann-Jordan constant* (NJ-constant in short) of a Banach (or normed) space  $X$ ,  $C_{NJ}(X)$ , is defined to be the smallest constant  $C$  for which

$$(7) \quad \frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C \quad (\forall (x, y) \neq (0, 0))$$

holds ([3]). We summarize some basic facts about NJ-constant.

**Proposition A.** (i)  $1 \leq C_{NJ}(X) \leq 2$  for any Banach space  $X$ ;  $C_{NJ}(X) = 1$  if and only if  $X$  is a Hilbert space (Jordan-von Neumann [4]).

(ii)  $C_{NJ}(X) < 2$  if and only if  $X$  is uniformly non-square (Takahashi-Kato [9]; see also [7]).

(iii)  $C_{NJ}(L_p) = C_{NJ}(\ell_p) = 2^{(2/t)-1}$ , where  $1 \leq p \leq \infty$ ,  $1/p + 1/p' = 1$  and  $t = \min\{p, p'\}$  (Clarkson [3]).

We determine and estimate the NJ-constant  $C_{NJ}(\|\cdot\|_\psi)$  of an absolute normalized norm  $\|\cdot\|_\psi$  by means of  $\psi$ . We easily have the following lemma.

**Lemma 1.** Let  $\varphi, \psi \in \Psi$  and  $\varphi \leq \psi$  for all  $0 \leq t \leq 1$ . Put

$$M = \max_{0 \leq t \leq 1} \frac{\psi(t)}{\varphi(t)}.$$

Then

$$\|\cdot\|_\varphi \leq \|\cdot\|_\psi \leq M \|\cdot\|_\varphi.$$

**Theorem 1** (Saito-Kato-Takahashi [10]). *Let  $\psi \in \Psi$ .*

(i) *Assume that  $\psi \geq \psi_2$ . Then*

$$(8) \quad C_{NJ}(\|\cdot\|_\psi) = \max_{0 \leq t \leq 1} \frac{\psi(t)^2}{\psi_2(t)^2}.$$

(ii) *Assume that  $\psi \leq \psi_2$ . Then*

$$(9) \quad C_{NJ}(\|\cdot\|_\psi) = \max_{0 \leq t \leq 1} \frac{\psi_2(t)^2}{\psi(t)^2}.$$

*Proof.* (i) Put  $M_1 = \max_{0 \leq t \leq 1} \psi(t)/\psi_2(t)$ . Then by Lemma 1

$$\begin{aligned} \|x+y\|_\psi^2 + \|x-y\|_\psi^2 &\leq M_1^2 (\|x+y\|_2^2 + \|x-y\|_2^2) \\ &= 2M_1^2 (\|x\|_2^2 + \|y\|_2^2) \\ &\leq 2M_1^2 (\|x\|_\psi^2 + \|y\|_\psi^2). \end{aligned}$$

Let  $M = \psi(t_1)/\psi_2(t_1)$  with some  $0 \leq t_1 \leq 1$ . Put  $x_1 = (1-t_1, 0)$ ,  $y_1 = (0, t_1)$ . Then we have

$$\|x_1 + y_1\|_\psi^2 + \|x_1 - y_1\|_\psi^2 = 2M_1^2 (\|x_1\|_\psi^2 + \|y_1\|_\psi^2),$$

which implies (8).

(ii) Put  $M_2 = \max_{0 \leq t \leq 1} \psi_2(t)/\psi(t)$ . Then in the same way as above we have

$$(10) \quad \|x+y\|_\psi^2 + \|x-y\|_\psi^2 \leq 2M_2^2 (\|x\|_\psi^2 + \|y\|_\psi^2).$$

Assume  $M_2 = \psi_2(t_2)/\psi(t_2)$  with some  $t_2$  ( $0 \leq t_2 \leq 1$ ). Then equality is attained in (10) with  $x_2 = (1-t_2, t_2)$  and  $y_2 = (1-t_2, -t_2)$ . Thus we have (9).

According to Theorem 1 the NJ-constant of  $\|\cdot\|_\psi$  does not depend on the shape of  $\psi$ . The following lemma is helpful to apply Theorem 1 ([10]).

**Lemma 2.** *Let  $\varphi(t) \geq \psi(t) > 0$  on  $[a, b]$ . Assume that  $\varphi - \psi$  has the maximum, resp.,  $\psi$  has the minimum at  $t = c$  in  $[a, b]$ . Then  $\varphi/\psi$  attains the maximum at  $t = c$ .*

**Corollary 1.** (i) *Let  $1 \leq p \leq \infty$  and  $1/p + 1/p' = 1$ . Let  $t = \min\{p, p'\}$ . Then*

$$(11) \quad C_{NJ}(\|\cdot\|_p) = 2^{(2/t)-1}.$$

*In particular,  $C_{NJ}(\|\cdot\|_1) = C_{NJ}(\|\cdot\|_\infty) = 2$  (Clarkson [3]).*

(ii) *Let  $2 \leq p < \infty$ . Let  $\|\cdot\|_{p,2}$  be the (Lorentz)  $\ell_{p,2}$ -norm;*

$$\|(z, w)\|_{p,2} = \{|z|^{*2} + 2^{(2/p)-1}|w|^{*2}\}^{1/2},$$

where  $\{|z|^*, |w|^*\}$  is the non-increasing rearrangement of  $\{|z|, |w|\}$ , that is,  $|z|^* \geq |w|^*$ . (Note that if  $p < 2$ ,  $\|\cdot\|_{p,2}$  is a quasi-norm; cf. [5, Proposition 1], [12, p.126]). Then

$$(12) \quad C_{NJ}(\|\cdot\|_{p,2}) = \frac{2}{1 + 2^{2/p-1}}.$$

*Proof.* (i) Let  $1 \leq p \leq 2$ . Then

$$\psi_2(t) \leq \psi_p(t) \leq 2^{(2/p)-1} \psi_2(t) \quad (0 \leq t \leq 1),$$

where the constant  $2^{(2/p)-1}$  is the best possible. Hence we have (11) by Theorem 1. For the case  $2 \leq p \leq \infty$  a similar argument works.

(ii) Clearly  $\|\cdot\|_{p,2} \in N_a$  and the corresponding convex function is given by

$$\psi_{p,2}(t) = \begin{cases} \{(1-t)^2 + 2^{2/p-1}t^2\}^{1/2} & \text{if } 0 \leq t \leq 1/2, \\ \{t^2 + 2^{2/p-1}(1-t)^2\}^{1/2} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Since  $\psi_{p,2} \leq \psi_2$ , and  $\psi_2/\psi_{p,2}$  is symmetric with respect to  $t = 1/2$ , we find the maximum of  $\psi_2^2/\psi_{p,2}^2$  in the interval  $[0, 1/2]$ . The difference  $\psi_2(t)^2 - \psi_{p,2}(t)^2 = (1 - 2^{2/p-1})t^2$  takes its maximum at  $t = 1/2$ , and  $\psi_{p,2}$  has the minimum at  $t = 1/2$ . Therefore by Lemma 2 we have

$$\max_{0 \leq t \leq 1} \frac{\psi_2(t)^2}{\psi_{p,2}(t)^2} = \frac{\psi_2(1/2)^2}{\psi_{p,2}(1/2)^2} = \frac{2}{1 + 2^{2/p-1}},$$

which implies (12) by Theorem 1.

*Remark 1.* The only known way to calculate NJ-constants needs Clarkson's inequalities (cf. [8]), whereas the above discussions to derive (11) and (12) do not require them.

For some further examples we refer to [10]. We have the following estimate for the general case.

**Theorem 2** (Saito-Kato-Takahashi [10]). *Let  $\psi \in \Psi$  and let*

$$(13) \quad M_1 = \max_{0 \leq t \leq 1} \frac{\psi(t)}{\psi_2(t)} \quad \text{and} \quad M_2 = \max_{0 \leq t \leq 1} \frac{\psi_2(t)}{\psi(t)}.$$

*Then*

$$(14) \quad \max\{M_1^2, M_2^2\} \leq C_{NJ}(\|\cdot\|_\psi) \leq M_1^2 M_2^2.$$

*Further we have*

$$(15) \quad 1 \leq \max\{M_1^2, M_2^2\} \leq M_1^2 M_2^2 \leq 2.$$

*Remark 2.* (i) From the proof of Theorem 2 we have that  $M_1^2 M_2^2 = 2$  if and only if  $\psi = \psi_1$  or  $\psi = \psi_\infty$ .

- (ii)  $\max\{M_1, M_2\} = 1$  if and only if  $\psi = \psi_2$ .  
 (iii)  $\max\{M_1, M_2\} = M_1 M_2$  if and only if  $\psi \geq \psi_2$  or  $\psi \leq \psi_2$ : In particular, Theorem 1 is derived from Theorem 2 and this fact.

By Remark 2 (i) we obtain

**Corollary 2.** *All absolute normalized norms on  $\mathbb{C}^2$  are uniformly non-square except the  $\ell_1$ - and  $\ell_\infty$ -norms.*

Theorem 1 gives a class of convex functions for which  $C_{NJ}(\|\cdot\|_\psi) = \max\{M_1^2, M_2^2\} = M_1^2 M_2^2$ . In [10] we gave a sufficient condition that  $C_{NJ}(\|\cdot\|_\psi) = M_1^2 M_2^2$  ( $\max\{M_1^2, M_2^2\} < C_{NJ}(\|\cdot\|_\psi)$ ), and that  $C_{NJ}(\|\cdot\|_\psi) < M_1^2 M_2^2$ , respectively. Some examples of absolute norms of non  $\ell_p$ -type are also given there.

### 3. Strict convexity and direct sums of Banach spaces

**Lemma B** ([2, p.36, Lemma 2]). *Let  $\|\cdot\| \in N_a$ . If  $|p| \leq |r|$  and  $|q| \leq |s|$ , then  $\|(p, q)\| \leq \|(r, s)\|$ . Further, if  $|p| < |r|$  and  $|q| < |s|$ , then  $\|(p, q)\| < \|(r, s)\|$ .*

Note that the latter assertion of Lemma B fails to hold if we replace the condition " $|p| < |r|$  and  $|q| < |s|$ " by " $|p| < |r|$  or  $|q| < |s|$ " (consider the  $\ell_\infty$ -norm).

**Lemma 3** ([11, Corollary 4]). *Let  $\psi \in \Psi$ . Then the following are equivalent.*

- (i) *If  $|p| \leq |r|$  and  $|q| < |s|$ , or  $|p| < |r|$  and  $|q| \leq |s|$ , then  $\|(p, q)\| < \|(r, s)\|$ .*  
 (ii)  *$\psi(t) > \psi_\infty(t)$  for all  $0 < t < 1$ .*

In particular we have the following corollary which is needed to obtain Theorems 3 and 4 below.

**Corollary 3** ([11]). *Let  $\psi \in \Psi$  be strictly convex. Let  $|p| \leq |r|$  and  $|q| \leq |s|$ , and let  $|p| < |r|$  or  $|q| < |s|$ . Then  $\|(p, q)\| < \|(r, s)\|$ .*

**Theorem 3** (Takahashi-Kato-Saito [11]). *Let  $\psi \in \Psi$ . Then  $\|\cdot\|_\psi$  is strictly convex if and only if  $\psi$  is strictly convex.*

Let  $X \oplus Y$  be the direct sum of Banach spaces  $X$  and  $Y$ . For any  $\psi \in \Psi$ , define

$$(16) \quad \|(x, y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi \quad \text{for } (x, y) \in X \oplus Y.$$

Let  $X \oplus_\psi Y$  denote  $X \oplus Y$  with the norm (16). Then we have

**Theorem 4** (Takahashi-Kato-Saito [11]). *Let  $X, Y$  be Banach spaces, and let  $\psi \in \Psi$ . Then  $X \oplus_\psi Y$  is strictly convex if and only if  $X, Y$  are strictly convex, and  $\psi$  is strictly convex.*

## References

- [1] B. Beauzamy, Introduction to Banach Spaces and Their Geometry, 2nd ed., North-Holland, Amsterdam-New York-Oxford, 1985.
- [2] F. F. Bonsall and J. Duncan, Numerical Ranges II, London Math. Soc. Lecture Note Ser. vol. 10, 1973.
- [3] J. A. Clarkson, The von Neumann-Jordan constant for the Lebesgue space, Ann. of Math. **38** (1937), 114-115.
- [4] P. Jordan and J. von Neumann, On inner products in linear metric spaces, Ann. of Math. **36** (1935), 719-723.
- [5] M. Kato, On Lorentz spaces  $\ell_{p,q}(E)$ , Hiroshima Math. J. **6** (1976), 73-93.
- [6] M. Kato, L. Maligranda and Y. Takahashi, On the Jordan-von Neumann constant and some related geometrical constants of Banach spaces, in preparation.
- [7] M. Kato and Y. Takahashi, On the von Neumann-Jordan constant for Banach spaces, Proc. Amer. Math. Soc. **125** (1997), 1055-1062.
- [8] M. Kato and Y. Takahashi, Von Neumann-Jordan constant for Lebesgue-Bochner spaces, J. Inequal. Appl. **2** (1998), 89-97.
- [9] Y. Takahashi and M. Kato, Von Neumann-Jordan constant and uniformly non-square Banach spaces, Nihonkai Math. J. **9** (1998), 155-169.
- [10] K.-S. Saito, M. Kato and Y. Takahashi, Von Neumann-Jordan constant of absolute normalized norms on  $\mathbb{C}^2$ , accepted in J. Math. Anal. Appl.
- [11] Y. Takahashi, M. Kato and K.-S. Saito, Strict convexity of absolute norms on  $\mathbb{C}^2$  and direct sums of Banach spaces, submitted.
- [12] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam-New York-Oxford, 1978.

*\*Department of Mathematics,  
Kyushu Institute of Technology,  
Kitakyushu 804-8550, Japan  
E-mail: katom@tobata.isc.kyutech.ac.jp*

*†Department of Mathematics,  
Faculty of Science,  
Niigata University,  
Niigata 950-2181, Japan  
E-mail: saito@math.sc.niigata-u.ac.jp*

*‡Department of System Engineering,  
Okayama Prefectural University,  
Soja 719-1197, Japan  
E-mail: takahasi@cse.oka-pu.ac.jp*